

THE STRUCTURE OF SURFACES MAPPING TO THE MODULI STACK OF CANONICALLY POLARIZED VARIETIES

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ABSTRACT. Generalizing the well-known Shafarevich hyperbolicity conjecture, it has been conjectured by Viehweg that a quasi-projective manifold that admits a generically finite morphism to the moduli stack of canonically polarized varieties is necessarily of log general type. Given a quasi-projective surface that maps to the moduli stack, we employ extension properties of logarithmic pluri-forms to establish a strong relationship between the moduli map and the minimal model program of the surface. As a result, we can describe the fibration induced by the moduli map quite explicitly. A refined affirmative answer to Viehweg's conjecture for families over surfaces follows as a corollary.

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1. INTRODUCTION AND MAIN RESULTS

1.A. Introduction. Let S° be a quasi-projective manifold that admits a morphism $\mu : S^\circ \rightarrow \mathfrak{M}$ to the moduli stack of canonically polarized varieties. Generalizing the classical Shafarevich hyperbolicity conjecture, [Sha63], Viehweg conjectured in [Vie01, 6.3] that S° is necessarily of log general type if μ is generically finite. Equivalently, if $f^\circ : X^\circ \rightarrow S^\circ$ is a smooth family of canonically polarized varieties, then S° is of log general type as soon as the variation of f° is maximal, i.e., $\text{Var}(f^\circ) = \dim S^\circ$. We refer to [KK05], for the relevant notions, for detailed references, and for a brief history of the problem.

Viehweg's conjecture was confirmed for 2-dimensional manifolds S° in [KK05]; see also [KS06]. Here, we complete the picture. The cornerstone of the proof is an extension theorem for logarithmic pluri-forms, Theorem 2.10. This theorem and its consequences

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are used to establish a strong relationship between the moduli map μ and the logarithmic minimal model program of the surface S° . This allows us to give a complete description of the moduli map in those cases where the variation cannot be maximal: the logarithmic minimal model program always ends with a fiber space, and the family comes from the base of this fibration, at least birationally and after suitable étale cover. Previous results and a refined affirmative answer to Viehweg's conjecture for families over surfaces follow as a corollary.

The proof of our main result is rather conceptual and completely independent of the argumentation of [KK05] which essentially relied on combinatorial arguments for curve arrangements on surfaces and on Keel-McKernan's solution to the Miyanishi conjecture in dimension 2, [KM99]. The present proof, besides giving a more complete picture, does not depend on the Keel-McKernan result at all. Many of the techniques introduced here generalize well to higher dimensions; most others at least conjecturally.

1.B. Main results. The following is the main result of this paper.

Theorem 1.1. *Let $f^\circ : X^\circ \rightarrow S^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective surface S° and S a compactification of S° such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. Assume that $\text{Var}(f^\circ) > 0$.*

Then $\kappa(S^\circ) \neq 0$. Furthermore, if $\kappa(S^\circ) < 2$, then any log minimal model program of the pair (S, D) will terminate at a fiber space, and the moduli map factors through the induced fibration of S° . More precisely, we have the following:

- (1.1.1) *If $\kappa(S^\circ) = -\infty$, then there exists an open set $U \subset S^\circ$ of the form $U = V \times \mathbb{A}^1$ such that $X^\circ|_U$ is the pull-back of a family over V . In particular, $\text{Var}(f^\circ) = 1$.*
- (1.1.2) *If $\kappa(S^\circ) = 1$, then there exists an open set $U \subset S^\circ$ and a Cartesian diagram of one of the following two types,*

$$\begin{array}{ccc}
 \tilde{U} & \xrightarrow[\text{étale}]{\gamma} & U \\
 \tilde{\pi} \downarrow & & \downarrow \pi \\
 \tilde{V} & \xrightarrow[\text{étale}]{} & V
 \end{array}
 \quad \text{or} \quad
 \begin{array}{ccc}
 \tilde{U} & \xrightarrow[\text{étale}]{\gamma} & U \\
 \tilde{\pi} \downarrow & & \downarrow \pi \\
 \tilde{V} & \xrightarrow[\text{étale}]{} & V
 \end{array}$$

$\tilde{\pi}$ elliptic fibration π elliptic fibration or $\tilde{\pi}$ smooth algebraic \mathbb{C}^* -bundle π smooth, algebraic \mathbb{C}^* -bundle

such that $X^\circ \times_U \tilde{U}$ is the pull-back of a family over \tilde{V} . In particular, $\text{Var}(f^\circ) = 1$.

Remark 1.2. Neither the compactification S nor the minimal model program discussed in Theorem 1.1 is unique. When running the minimal model program, one often needs to choose the extremal ray that is to be contracted.

A somewhat more precise version of Viehweg's conjecture for surfaces also follows as an immediate consequence of Theorem 1.1, cf. [KK05, Conjecture 1.6].

Corollary 1.3 (Viehweg's conjecture for surfaces, [KK05, Thm. 1.4]). *Let $f^\circ : X^\circ \rightarrow S^\circ$ be a smooth projective family of canonically polarized varieties over a quasi-projective surface S° . Then either $\kappa(S^\circ) = -\infty$ and $\text{Var}(f^\circ) < \dim S^\circ$, or $\text{Var}(f^\circ) \leq \kappa(S^\circ)$. \square*

1.C. Conventions and notation. Throughout the present paper we work over the complex number field. When dealing with sheaves that are not necessarily locally free, we frequently use square brackets to indicate taking the reflexive hull.

Notation 1.4. Let Y be a normal variety and \mathcal{A} a coherent sheaf of \mathcal{O}_Y -modules. Let $n \in \mathbb{N}$ and set $\mathcal{A}^{[n]} := (\mathcal{A}^{\otimes n})^{**}$, $\text{Sym}^{[n]} \mathcal{A} := (\text{Sym}^n \mathcal{A})^{**}$, etc. Likewise, for a morphism $f : X \rightarrow Y$ of normal varieties, set $f^{[*]} \mathcal{A} := (f^* \mathcal{A})^{**}$.

We will later discuss the Kodaira dimension of singular pairs and the Kodaira-Iitaka dimension of reflexive sheaves on normal spaces. Since this is perhaps not quite standard, we recall the definition here.

Notation 1.5. Let Y be a normal projective variety and \mathcal{A} a reflexive sheaf of rank one on Y . If $h^0(Y, \mathcal{A}^{[n]}) = 0$ for all $n \in \mathbb{N}$, then we say that \mathcal{A} has Kodaira-Iitaka dimension $\kappa(\mathcal{A}) := -\infty$. Otherwise, recall that the restriction of \mathcal{A} to the smooth locus of Y is locally free and consider the rational mapping

$$\phi_n : Y \dashrightarrow \mathbb{P}(H^0(Y, \mathcal{A}^{[n]})^*).$$

The Kodaira-Iitaka dimension of \mathcal{A} is then defined as

$$\kappa(\mathcal{A}) := \max_{n \in \mathbb{N}} (\dim \overline{\phi_n(Y)}).$$

If $D \subset Y$ is an effective Weil divisor, define the Kodaira dimension of the pair (Y, D) as $\kappa(Y, D) := \kappa(\mathcal{O}_Y(K_Y + D))$. If Y is smooth and D is a simple normal crossing divisor, define the Kodaira dimension of the complement $Y^\circ = Y \setminus D$ as $\kappa(Y^\circ) := \kappa(Y, D)$. Recall, that $\kappa(Y^\circ)$ is independent of the choice of the compactification Y .

1.D. Outline of proof, outline of paper. The technical core of this paper is the extension result for pluri-log forms, formulated in Theorems 2.10 and 2.15 of Section 2. In essence, it states the following: If (S, D) is a pair of a smooth surface and a reduced divisor with simple normal crossings, (S_λ, D_λ) a log-minimal model, and $\mathcal{A}_\lambda \subset \text{Sym}^{[n]} \Omega_{S_\lambda}^1(\log D_\lambda)$ any rank-one reflexive sheaf of pluri-log forms, then \mathcal{A}_λ pulls back to a reflexive sheaf of pluri-log forms in $\text{Sym}^n \Omega_S^1(\log D')$, where D' is a divisor that is only slightly larger than D . Under the conditions of Theorem 1.1, a fundamental result of Viehweg and Zuo asserts that a rank-one reflexive subsheaf $\mathcal{A}_\lambda \subset \text{Sym}^{[n]} \Omega_{S_\lambda}^1(\log D_\lambda)$ of positive Kodaira-Iitaka dimension always exists.

The extension theorem is applied, e.g., in Section 3, in order to give a criterion that is later used to show the fiber space structure of certain minimal models. For an idea of the statement and its proof, consider the setup of Theorem 1.1 in the simplest case where $\kappa(S^\circ) = -\infty$. The log-minimal model (S_λ, D_λ) will then either be log-Fano of Picard-number one, or a Mori-Fano fiber space. To show that (S_λ, D_λ) is a Mori-Fano fiber space, we argue by contradiction and assume that $\rho(S_\lambda) = 1$. Using this assumption and the existence of \mathcal{A}_λ , an analysis of the stability of $\Omega_{S_\lambda}^{[1]}(\log D_\lambda)$ yields the existence of a \mathbb{Q} -ample rank-one subsheaf $\mathcal{B}_\lambda \subset \Omega_{S_\lambda}^{[1]}(\log D_\lambda)$. The Extension Theorem will then show the existence of a big invertible subsheaf $\mathcal{B} \subset \Omega_S^1(\log D')$. This, however, contradicts the well-known Bogomolov-Sommese vanishing result, and the existence of a fiber space structure is shown.

The argumentation in case $\kappa(S^\circ) = 0$ follows a similar outline, but is technically much more involved. Section 4 gathers results that are particular to the case $\kappa = 0$, work in any dimension and may be of independent interest. The detailed description of the moduli map for fiber spaces is done in a unified framework in Section 5.

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PART I. TECHNIQUES

2. EXTENDING PLURI-FORMS OVER SUBVARIETIES OF CODIMENSION ONE

If X is a surface, $E \subset X$ a (-1) -curve and $\omega \in H^0(X, \Omega_X^p(*E))$ a p -form that is allowed to have arbitrary poles along E , then an elementary computation shows that ω is in fact everywhere regular on X , i.e., $\omega \in H^0(X, \Omega_X^p)$.

Much of the argumentation in this paper is based on the observation that a slightly weaker result also holds for pluri-log forms, and for somewhat larger classes of divisors. We refer to [vSS85, Fle88] for more general extension results that apply to holomorphic p -forms.

2.A. Notation and standard facts about logarithmic differentials. We introduce notation and recall two standard facts before stating and proving the extension result in Section 2.C below. These make sense and will be used both in the algebraic and in the analytic category. We refer to [Lit82, Chapt. 11c] and [Del70, Chap. 3] for details and proofs.

Definition 2.1. A reduced pair (Z, Δ) consists of a normal variety Z and a reduced, but not necessarily irreducible Weil divisor $\Delta \subset Z$. A morphism of reduced pairs $\gamma : (\tilde{Z}, \tilde{\Delta}) \rightarrow (Z, \Delta)$ is a morphism $\gamma : \tilde{Z} \rightarrow Z$ such that $\gamma^{-1}(\Delta) = \tilde{\Delta}$ set-theoretically.

A reduced pair is called log smooth if Z is smooth and Δ has simple normal crossings. Given a reduced pair (Z, Δ) , let $(Z, \Delta)_{\text{reg}}$ be the maximal open set of Z where (Z, Δ) is log-smooth, and let $(Z, \Delta)_{\text{sing}}$ be its complement, with the structure of a reduced subscheme. By a log-resolution of (Z, Δ) we will mean a birational morphism of pairs $\gamma : (\tilde{Z}, \tilde{\Delta}) \rightarrow (Z, \Delta)$ where $(\tilde{Z}, \tilde{\Delta})$ is log-smooth, and γ is isomorphic along $(Z, \Delta)_{\text{reg}}$.

Fact 2.2 ([Hir62]). Let (Z, Δ) be a reduced pair. Then a log-resolution exists. If (Z, Δ) is log-smooth, then the sheaf of log-differentials $\Omega_Z^1(\log \Delta)$ is locally free. \square

Fact 2.3. Let $\gamma : (\tilde{Z}, \tilde{\Delta}) \rightarrow (Z, \Delta)$ be a morphism of log-smooth reduced pairs, $U \subseteq Z$ an open set and $\tilde{U} = \gamma^{-1}(U)$. Then there exists a natural pull-back map of log-forms

$$\gamma^* : H^0(U, \Omega_Z^1(\log \Delta)) \rightarrow H^0(\tilde{U}, \Omega_{\tilde{Z}}^1(\log \tilde{\Delta})).$$

and an associated sheaf morphism

$$d\gamma : \gamma^*(\Omega_Z^1(\log \Delta)) \rightarrow \Omega_{\tilde{Z}}^1(\log \tilde{\Delta}).$$

If γ is finite and unramified over $Z \setminus \Delta$, then $d\gamma$ is isomorphic. \square

Remark 2.3.1. The pull-back morphism also gives a pull-back of pluri-log forms,

$$\gamma^* : H^0(Z, \text{Sym}^n \Omega_Z^1(\log \Delta)) \rightarrow H^0(\tilde{Z}, \text{Sym}^n \Omega_{\tilde{Z}}^1(\log \tilde{\Delta})),$$

that obviously extends to a pull-back of rational forms.

We state one immediate consequence of Fact 2.3 for future reference.

Corollary 2.4. Under the conditions of Fact 2.3, assume that γ is a finite morphism which is unramified over $Z \setminus \Delta$. Let $E \subset Z$ be an effective divisor and $\sigma \in H^0(Z, \text{Sym}^n(\Omega_Z^1(\log \Delta))(*E))$ a pluri-log form that might have poles along E .

Then σ has no poles along E , i.e., $\sigma \in H^0(Z, \text{Sym}^n \Omega_Z^1(\log \Delta))$ if and only if $\gamma^*(\sigma)$ has no poles along $\gamma^{-1}E$, i.e., $\gamma^*(\sigma) \in H^0(\tilde{Z}, \text{Sym}^n \Omega_{\tilde{Z}}^1(\log \tilde{\Delta}))$. \square

Notation 2.5. In the setup of Corollary 2.4, we say that “ σ has poles as a pluri-log form if and only if $\gamma^*(\sigma)$ has poles as a pluri-log form”.

2.B. Finitely dominated pairs. The formulation of the main extension result in Theorem 2.10 uses the following notion, which slightly generalizes quotient singularities.

Definition 2.6. A reduced pair (Z, Δ) is said to be finitely dominated by smooth analytic pairs if for any point $z \in Z$, there exists an analytic neighborhood U of z and a finite, surjective morphism of reduced pairs $(\tilde{U}, \tilde{\Delta}) \rightarrow (U, \Delta \cap U)$ where $(\tilde{U}, \tilde{\Delta})$ is log-smooth.

Surface singularities that appear in certain variants of the minimal model program are often finitely dominated by smooth analytic pairs. In the rest of this subsection we discuss a class of examples that will become important later.

Definition 2.7. A reduced pair (Z, Δ) is called dlc if (Z, Δ) is lc and $Z \setminus \Delta$ is lt.

Example 2.8. It follows immediately from the definition that *dlt* pairs are *dlc*. For a less obvious example, let Z be the cone over a conic and Δ the union of two rays through the vertex. Then (Z, Δ) is *dlc*, but not *dlt*.

Lemma 2.9. *Let (Z, Δ) be a dlc pair of dimension 2. Then (Z, Δ) is finitely dominated by smooth analytic pairs. In particular, if (Z, Δ) is dlt, then it is finitely dominated by smooth analytic pairs.*

Proof. Let $z \in (Z, \Delta)_{\text{sing}}$ be an arbitrary singular point. If $z \notin \Delta$, then the statement follows from [KM98, 4.18]. We can thus assume without loss of generality for the remainder of the proof that $z \in \Delta$.

To continue, observe that for any rational number $0 < \varepsilon < 1$, the non-reduced pair $(Z, (1 - \varepsilon)\Delta)$ is *numerically dlt*; see [KM98, 4.1] for the definition and use [KM98, 3.41] for an explicit discrepancy computation. By [KM98, 4.11], Z is then \mathbb{Q} -factorial. Using \mathbb{Q} -factoriality, we can then choose a sufficiently small Zariski neighborhood U of z and consider the index-one cover for $\Delta \cap U$. This gives a finite morphism of pairs $\gamma : (\tilde{U}, \tilde{\Delta}) \rightarrow (U, \Delta \cap U)$, where the morphism γ is branched only over the singularities of U , and where $\tilde{\Delta} = \gamma^*(\Delta \cap U)$ is Cartier—see [KM98, 5.19] for the construction. Choose any point $\tilde{z} \in \gamma^{-1}(z)$. Since discrepancies only increase under taking finite covers, [KM98, 5.20], the pair $(\tilde{U}, \tilde{\Delta})$ will again be *dlc*. In particular, it suffices to prove the claim for a neighborhood of \tilde{z} in $(\tilde{U}, \tilde{\Delta})$. We can thus assume without loss of generality that $z \in \Delta$ and that Δ is Cartier in our original setup.

Next, we claim that (Z, \emptyset) is canonical at z . In fact, let E be any divisor centered above z , as in [KM98, 2.24]. Since $z \in \Delta$, and since Δ is Cartier, the pull-back of Δ to any resolution where E appears will contain E with multiplicity at least 1. In particular, we have the following inequality for the log discrepancies: $0 \leq a(E, Z, \Delta) + 1 \leq a(E, Z, \emptyset)$. This shows that (Z, \emptyset) is canonical at z as claimed.

By [KM98, 4.20–21], Z has a Du Val quotient singularity at z . Again replacing Z by a finite cover of a suitable neighborhood of z , and replacing z by its preimage in the covering space, we can henceforth assume without loss of generality that Z is smooth. But then the claim follows from [KM98, 4.15]. \square

2.C. The extension theorem for finitely dominated pairs. The following is the main result of the present section. It asserts that any log form defined outside of a divisor E can be extended to the whole space if E contracts to a singularity which is finitely dominated by a smooth analytic pair. Theorem 2.10 holds in arbitrary dimension.

Theorem 2.10 (Extension Theorem for finitely dominated singularities). *Let (Z, Δ) be a reduced pair in the sense of Definition 2.1, and assume that (Z, Δ) is finitely dominated by smooth analytic pairs. Let $\psi : (Y, \Gamma) \rightarrow (Z, \Delta)$ be a log-resolution, $E_\Gamma \subset Y$ the union of the ψ -exceptional divisors that are not contained in Γ , and $n \in \mathbb{N}$. Then $\psi_* \text{Sym}^n \Omega_{Y/\Gamma}^1(\log(\Gamma + E_\Gamma))$ is reflexive.*

Remark 2.10.1. Let $E \subset Y$ be the exceptional set of ψ . Then Theorem 2.10 is equivalent to the statement that for any open set $U \subset Z$ with preimage $V := \psi^{-1}(U)$ and any form $\sigma \in H^0(V \setminus E, \text{Sym}^n \Omega_{V \setminus E}^1(\log \Gamma))$ defined outside the ψ -exceptional set $E \cap V$, the form σ extends to a form $\tilde{\sigma} \in H^0(V, \text{Sym}^n \Omega_V^1(\log(\Gamma + E_\Gamma)))$ on all of V . Hence the name “extension theorem”.

Remark 2.10.2. For an example in the simple case where $\Delta = \emptyset$, let Y be the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$, and let E be the zero-section. It is not very difficult to write down a pluri-log form

$$\sigma \in H^0(Y, \text{Sym}^2 \Omega_Y^1(\log E)) \setminus H^0(Y, \text{Sym}^2 \Omega_Y^1).$$

Because E contracts to a quotient singularity, this example shows that Theorem 2.10 holds only for log-differentials, and that the boundary given in Theorem 2.10 is the smallest possible.

In order to construct σ , consider the standard coordinate cover of Y with open sets $U_{1,2} \simeq \mathbb{A}^2$, where U_i carries coordinates x_i, y_i and coordinate change is given as

$$\phi_{1,2} : (x_1, y_1) \mapsto (x_2, y_2) = (x_1^{-1}, x_1^2 y_1).$$

In these coordinates the bundle map $U_i \rightarrow \mathbb{P}^1$ is given as $(x_i, y_i) \rightarrow x_i$, and the zero-section E is given as $E \cap U_i = \{y_i = 0\}$. Now take

$$\sigma_2 := y_2^{-1} (dy_2)^2 \in (\mathrm{Sym}^2(\Omega_Y^1(\log E)))(U_2)$$

and observe that $\phi_{1,2}^*(\sigma)$ extends to a form in $(\mathrm{Sym}^2(\Omega_Y^1(\log E)))(U_1)$.

Proof of Theorem 2.10. Assume that we are given an open set U and a form σ as in Remark 2.10.1. Since the extension problem is local on Z in the analytic topology, we can shrink Z and assume without loss of generality that there exists a finite, surjective morphism $\tilde{\gamma} : (\tilde{Z}, \tilde{\Delta}) \rightarrow (Z, \Delta)$ from a smooth pair $(\tilde{Z}, \tilde{\Delta})$.

Let \tilde{Y} be the normalization of $Y \times_{\tilde{Z}} Z$ and $\tilde{\Gamma} \subset \tilde{Y}$ the reduced preimage of Γ . Then we obtain a commutative diagram of surjective morphisms of pairs as follows,

$$\begin{array}{ccc} (\tilde{Y}, \tilde{\Gamma}) & \xrightarrow{\tilde{\gamma}, \text{finite}} & (Y, \Gamma) \\ \tilde{\psi} \downarrow \text{contracts } \tilde{E} & & \downarrow \psi \text{ log-resolution, contracts } E \\ (\tilde{Z}, \tilde{\Delta}) & \xrightarrow{\gamma, \text{finite}} & (Z, \Delta) \end{array}$$

where $\tilde{E} := (\tilde{\gamma}^{-1}(E))_{\mathrm{red}} = ((\gamma \circ \tilde{\psi})^{-1}(Z, \Delta)_{\mathrm{sing}})_{\mathrm{red}}$ is the exceptional set of the morphism $\tilde{\psi}$. Let $B \subset Z$ be the branch divisor of γ , i.e., the minimal codimension-1 set such that $\gamma|_{\tilde{Z} \setminus \gamma^{-1}(B)}$ is étale in codimension one. Let $\psi_*^{-1}(B) \subset Y$ be its strict transform. Finally, set

$$Y^0 := Y \setminus (\psi_*^{-1}(B) \cup \tilde{\gamma}((\tilde{Y}, \tilde{\Gamma})_{\mathrm{sing}})).$$

The set Y^0 is then the maximal open subset of $Y \setminus \psi_*^{-1}(B)$ such that $\tilde{Y}^0 := \tilde{\gamma}^{-1}(Y^0)$ is contained in the log-smooth locus of $(\tilde{Y}, \tilde{\Gamma})$. We will use two of its main properties explicitly. These are contained in the following Claims.

Claim 2.10.3. The complement $Y \setminus Y^0$ intersects the ψ -exceptional set E only in a set of codimension $\mathrm{codim}_Y(E \setminus Y^0) \geq 2$.

Proof. We need to show that

$$\begin{aligned} 2 &\leq \mathrm{codim}_Y((\psi_*^{-1}(B) \cup \tilde{\gamma}((\tilde{Y}, \tilde{\Gamma})_{\mathrm{sing}})) \cap E) \\ &= \min\{\mathrm{codim}_Y(\psi_*^{-1}(B) \cap E), \mathrm{codim}_Y(\gamma((\tilde{Y}, \tilde{\Gamma})_{\mathrm{sing}}) \cap E)\}. \end{aligned}$$

Since \tilde{Y} is normal, the log-singular locus $(\tilde{Y}, \tilde{\Gamma})_{\mathrm{sing}}$ has codimension at least 2. Since $\tilde{\gamma}$ is finite, this gives $\mathrm{codim}_Y \tilde{\gamma}((\tilde{Y}, \tilde{\Gamma})_{\mathrm{sing}}) \geq 2$. It is also clear that $\psi_*^{-1}(B)$ and E have no common component, so $\mathrm{codim}_Y \psi_*^{-1}(B) \cap E \geq 2$. \square

Claim 2.10.4. The morphism $\tilde{\gamma}|_{\tilde{Y}^0}$ is étale outside of $E \cap Y^0$.

Proof. By construction, $\tilde{\gamma}$ is étale outside of $E \cup \psi_*^{-1}(B)$. \square

To prove Theorem 2.10, we need to show that σ extends to all of Y as a pluri-log form, i.e., that the associated section

$$\bar{\sigma} \in H^0(Y, (\mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma)))(*E))$$

has no poles along E as a pluri-log form. Since $\bar{\sigma}$ certainly has no poles outside of E , and since $\mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma))$ is locally free, Claim 2.10.3 implies that it suffices to show that the restriction $\bar{\sigma}|_{Y^0}$ has no poles along $E \cap Y^0$ as a pluri-log form. In particular, $\bar{\sigma} \in H^0(Y, \mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma)))$.

By Corollary 2.4 and Claim 2.10.4, it suffices to show that the pull-back $\tilde{\gamma}^*(\bar{\sigma})|_{\tilde{Y}^0}$ does not have any poles along $\tilde{Y}^0 \cap \tilde{E}$ as a pluri-log form. For that, recall that ψ is an isomorphism over $(Z, \Delta)_{\text{reg}}$. Hence the form σ gives rise to a form

$$\tau \in H^0(Z, \text{Sym}^{[n]} \Omega_Z^1(\log \Delta)).$$

Since $(\tilde{Z}, \tilde{\Delta})$ is log-smooth, Fact 2.3 asserts that the pull-back of τ extends to a pluri-log form $\tilde{\tau} \in H^0(\tilde{Z}, \text{Sym}^n \Omega_{\tilde{Z}}^1(\log \tilde{\Delta}))$ on all of \tilde{Z} . The pull-back to \tilde{Y}^0 ,

$$(2.10.5) \quad \hat{\tau} := \tilde{\psi}^*(\tilde{\tau}) \in H^0(\tilde{Y}^0, \text{Sym}^n \Omega_{\tilde{Y}}^1(\log \tilde{\Gamma})|_{\tilde{Y}^0}),$$

is then a pluri-log form on \tilde{Y}^0 without poles that agrees with $\tilde{\gamma}^*(\sigma)$ outside \tilde{E} . This form necessarily equals $(\tilde{\gamma}|_{\tilde{Y}^0})^*(\bar{\sigma})$, which then does not have any poles along $\tilde{Y}^0 \cap \tilde{E}$, as asserted. Theorem 2.10 follows. \square

Corollary 2.11. *Under the conditions of Theorem 2.10 we obtain an embedding*

$$\psi^{[*]} \text{Sym}^{[n]} \Omega_Z^1(\log \Delta) \hookrightarrow \text{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma)).$$

Proof. As ψ induces an isomorphism $Y \setminus E \simeq Z \setminus \psi(E)$, Theorem 2.10 implies that

$$\text{Sym}^{[m]} \Omega_Z^1(\log \Delta) \simeq \psi_* \text{Sym}^m \Omega_Y^1(\log(\Gamma + E_\Gamma))$$

and hence we obtain that there exists a morphism

$$\psi^* \text{Sym}^{[m]} \Omega_Z^1(\log \Delta) \simeq \psi^* \psi_* \text{Sym}^m \Omega_Y^1(\log(\Gamma + E_\Gamma)) \rightarrow \text{Sym}^m \Omega_Y^1(\log(\Gamma + E_\Gamma)),$$

which is an isomorphism, in particular an embedding, on $Y \setminus E$. This remains true after taking the double dual of these sheaves. Therefore the kernel of the map $\psi^{[*]} \text{Sym}^{[m]} \Omega_Z^1(\log \Delta) \rightarrow \text{Sym}^m \Omega_Y^1(\log(\Gamma + E_\Gamma))$ is a torsion sheaf and the fact that $\psi^{[*]} \text{Sym}^{[m]} \Omega_Z^1(\log \Delta)$ is torsion-free implies the statement. \square

2.D. Extensions of Viehweg-Zuo sheaves. We believe that the conclusion of Theorem 2.10 holds for a larger class of singularities than those that we need to discuss here. Thus it makes sense to introduce the following notation.

Definition 2.12. *Let (Z, Δ) be a reduced pair in the sense of Definition 2.1. Then we will say that the extension theorem holds for (Z, Δ) if for any log-resolution $\psi : (Y, \Gamma) \rightarrow (Z, \Delta)$, the sheaf $\psi_* \text{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma))$ is reflexive, where E_Γ denotes the union of the ψ -exceptional divisors that are not contained in Γ , and $n \in \mathbb{N}$ is arbitrary.*

Example 2.13. Example 2.9 and Theorem 2.10 imply that the extension theorem holds for dlc surface pairs.

We will later consider log-smooth reduced pairs (Z, Δ) and morphisms $f : Y \rightarrow Z$ whose restriction to $Z \setminus \Delta$ is a smooth family of canonically polarized varieties. If f has positive variation, $\text{Var}(f) > 0$, then Viehweg and Zuo have shown in [VZ02, Thm. 1.4] that there exists a positive number n and an invertible subsheaf $\mathcal{A} \subset \text{Sym}^n \Omega_Z^1(\log \Delta)$ of Kodaira-Iitaka dimension $\kappa(\mathcal{A}) \geq \text{Var}(f)$. We call this a *Viehweg-Zuo sheaf* on (Z, Δ) . More generally and more precisely, we use the following definition.

Definition 2.14. *Let (Z, Δ) be a reduced pair. A reflexive sheaf \mathcal{A} of rank 1 is called a Viehweg-Zuo sheaf if for some $n \in \mathbb{N}$ there exists an embedding $\mathcal{A} \subset \text{Sym}^{[n]} \Omega_Z^1(\log \Delta)$.*

The extension theorem will be used later to pull-back Viehweg-Zuo sheaves to log resolutions. The following Theorem shows how this is done.

Theorem 2.15 (Extension of Viehweg-Zuo sheaves). *Let (Z, Δ) be a reduced pair for which the extension theorem holds. Using the setup of Definition 2.12, assume that there exists a Viehweg-Zuo sheaf \mathcal{A} with inclusion $\iota : \mathcal{A} \rightarrow \text{Sym}^{[n]} \Omega_Z^1(\log \Delta)$. Then there*

exists an invertible Viehweg-Zuo sheaf $\mathcal{C} \subset \mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma))$ with the following property: Let $m \in \mathbb{N}$ and

$$\iota^{[m]} : \mathcal{A}^{[m]} \rightarrow \mathrm{Sym}^{[m \cdot n]} \Omega_Z^1(\log \Delta)$$

the associated morphism of reflexive powers. Then $\iota^{[m]}$ pulls back to give a sheaf morphism that factors through $\mathcal{C}^{\otimes m}$,

$$\bar{\iota}^{[m]} : \psi^{[*]} \mathcal{A}^{[m]} \hookrightarrow \mathcal{C}^{\otimes m} \subset \mathrm{Sym}^{m \cdot n} \Omega_Y^1(\log(\Gamma + E_\Gamma)).$$

Proof. By Corollary 2.11, $\psi^{[*]} \mathcal{A}$ embeds into $\mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma))$. Let $\mathcal{C} \subset \mathrm{Sym}^n \Omega_Y^1(\log(\Gamma + E_\Gamma))$ be the saturation of the image, which is automatically reflexive by [OSS80, Lem. 1.1.16 on p. 158]. By [OSS80, Lem. 1.1.15 on p. 154], \mathcal{C} is then invertible as desired. Further observe that for any $m \in \mathbb{N}$, the subsheaf $\mathcal{C}^{\otimes m} \subset \mathrm{Sym}^{m \cdot n} \Omega_Y^1(\log(\Gamma + E_\Gamma))$ is likewise saturated. Again, by Corollary 2.11, there exists an embedding,

$$\bar{\iota}^{[m]} : \psi^{[*]} \mathcal{A}^{[m]} \hookrightarrow \mathrm{Sym}^{m \cdot n} \Omega_Y^1(\log(\Gamma + E_\Gamma)).$$

It is easy to see that $\bar{\iota}^{[m]}$ factors through $\mathcal{C}^{\otimes m}$ as it does so on the open set where ψ is isomorphic, and because $\mathcal{C}^{\otimes m}$ is saturated. \square

Remark 2.16. Under the conditions of Theorem 2.15, observe that the Kodaira-Iitaka dimension of \mathcal{C} is at least the Kodaira-Iitaka dimension of \mathcal{A} , i.e., $\kappa(\mathcal{C}) \geq \kappa(\mathcal{A})$.

3. VIEHWEG-ZUO SHEAVES ON LOG MINIMAL MODELS

The existence of a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension clearly has consequences for the geometry of the underlying space. The following theorem will later be used to show that a given pair is a Mori-Fano fiber space. This will turn out to be a key step in the proof of our main results. We refer to Definition 2.14 for the notion of a Viehweg-Zuo sheaf.

Theorem 3.1. *Let (Z, Δ) be a reduced pair such that Z is a normal and \mathbb{Q} -factorial surface. Assume that the following holds:*

- (3.1.1) *there exists a Viehweg-Zuo sheaf $\mathcal{A} \subset \mathrm{Sym}^{[n]} \Omega_Z^1(\log \Delta)$ of positive Kodaira-Iitaka dimension,*
- (3.1.2) *the extension theorem holds for (Z, Δ) , and*
- (3.1.3) *the anti-log-canonical divisor $-(K_Z + \Delta)$ is nef.*

Then $\rho(Z) > 1$.

Proof. We argue by contradiction and assume that $\rho(Z) = 1$. Let $C \subset Z$ be a general complete intersection curve. Since C is general, it avoids the singular locus $(Z, \Delta)_{\mathrm{sing}}$. By (3.1.3), the restriction $\Omega_Z^1(\log \Delta)|_C$ is a vector bundle of non-positive degree,

$$(3.2.1) \quad \deg \Omega_Z^1(\log \Delta)|_C = (K_Z + \Delta) \cdot C \leq 0.$$

We claim that the restriction $\Omega_Z^1(\log \Delta)|_C$ is not anti-nef, i.e., that the dual vector bundle $\Omega_Z^1(\log \Delta)^*|_C$ is not nef. In particular, we claim that $\Omega_Z^1(\log \Delta)|_C$ admits a subsheaf of positive degree. Indeed, if $\Omega_Z^1(\log \Delta)|_C$ were anti-nef, then none of its symmetric products $\mathrm{Sym}^n \Omega_Z^1(\log \Delta)|_C$ could contain a subsheaf of positive degree. However, since C is general, the restriction of the Viehweg-Zuo sheaf to C is a locally free subsheaf $\mathcal{A}|_C \subset \mathrm{Sym}^n \Omega_Z^1(\log \Delta)|_C$ of positive Kodaira-Iitaka dimension, and hence of positive degree. This proves the claim.

As a consequence of the claim and of Equation (3.2.1), we obtain that $\Omega_Z^{[1]}(\log \Delta)$ is not semi-stable and if $\mathcal{B} \subset \Omega_Z^{[1]}(\log \Delta)$ denotes the maximal destabilizing subsheaf, its slope $\mu(\mathcal{B})$ is positive. The assumption that $\rho(Z) = 1$ and \mathbb{Q} -factoriality then guarantees that \mathcal{B} is \mathbb{Q} -ample. In particular, its Kodaira-Iitaka dimension is maximal, $\kappa(\mathcal{B}) = 2$.

Now consider a log-resolution $\psi : (Y, \Gamma) \rightarrow (Z, \Delta)$ as in Definition 2.12. The Extension Theorem for Viehweg-Zuo sheaves, Theorem 2.15, Remark 2.16, and the assumption that $\rho(Z) = 1$ guarantee the existence of a Viehweg-Zuo sheaf $\mathcal{C} \subset \Omega_Y^1(\log \Gamma + E_\Gamma)$ of Kodaira-Iitaka dimension $\kappa(\mathcal{C}) = 2$. As there are no symmetric tensors involved, this contradicts the Bogomolov-Sommese vanishing theorem, [EV92, Cor. 6.9]. \square

4. GLOBAL INDEX-ONE COVERS FOR VARIETIES OF LOGARITHMIC KODAIRA DIMENSION 0

In this section, we consider a smooth pair (Y, D) of Kodaira dimension 0, go to a minimal model and take the global index-one cover. If (Y, D) carries a Viehweg-Zuo sheaf $\mathcal{A} \subset \text{Sym}^n \Omega_Y^1(\log D)$ of positive Kodaira-Iitaka dimension, then we show that the cover is uniruled and that its boundary is not empty. All results of this section hold in arbitrary dimension.

4.A. Construction of the cover. First we briefly recall the main properties of the index-one cover, as described in [KM98, 2.52] or [Rei87, Sect. 3.6f].

Proposition 4.1. *Let (Y, D) be a reduced, log-smooth pair of dimension $\dim Y \geq 2$ and Kodaira dimension $\kappa(Y, D) = 0$. Assume that there exists a birational map $\lambda : Y \dashrightarrow Y_\lambda$ to a normal variety Y_λ , such that the following holds.*

- (4.1.1) *The inverse λ^{-1} does not contract any divisor.*
- (4.1.2) *(Y_λ, D_λ) is a log minimal model of (Y, D) , where D_λ denotes the cycle-theoretic image of D .*
- (4.1.3) *The log abundance conjecture holds for (Y_λ, D_λ) .*

Then there exists a diagram

$$\begin{array}{ccc}
 Y & & \tilde{Y} \\
 \downarrow \lambda & & \downarrow \tilde{\lambda} \\
 Y_\lambda & \xleftarrow[\text{finite, étale where } Y_\lambda \text{ is smooth}]{\gamma, \text{ index-one cover}} & \tilde{Y}_\lambda
 \end{array}$$

$\tilde{\lambda}$ is labeled "log resolution" next to the arrow from \tilde{Y} to \tilde{Y}_λ .

with the following properties.

- (4.1.4) *If $\tilde{D}_\lambda := \gamma^*(D_\lambda)$, then $K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda$ is Cartier with $\mathcal{O}_{\tilde{Y}_\lambda}(K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda) \simeq \mathcal{O}_{\tilde{Y}_\lambda}$.*
- (4.1.5) *The pair $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is dlt. If $y \in \tilde{Y}_\lambda$ is a point where $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is not log-smooth, then $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is canonical at y .*
- (4.1.6) *If $\tilde{D} = \tilde{\lambda}^*(\tilde{D}_\lambda)_{\text{red}}$, then $\kappa(\tilde{Y}, \tilde{D}) = 0$.*

For the reader's convenience, we recall a few notions of higher dimensional geometry used in the formulation of Proposition 4.1.

Notation 4.2. A log minimal model is a dlt pair (Y_λ, D_λ) where Y_λ is \mathbb{Q} -factorial and where $K_{Y_\lambda} + D_\lambda$ is nef, cf. [KM98, 3.29–31]. If (Y_λ, D_λ) is a log minimal model and has Kodaira dimension $\kappa(Y_\lambda, D_\lambda) = 0$, we say that *the log abundance conjecture holds for (Y_λ, D_λ)* if there exists a number $k \in \mathbb{N}^+$ such that $k \cdot (K_{Y_\lambda} + D_\lambda)$ is Cartier and $\mathcal{O}_{Y_\lambda}(k \cdot (K_{Y_\lambda} + D_\lambda)) \simeq \mathcal{O}_{Y_\lambda}$, cf. [KM98, 3.12].

Remark 4.3. The existence of log minimal models and log abundance for minimal models is currently known for $\dim Y \leq 3$, see [KM98, 3.13] for references concerning abundance. Both are expected to hold in any dimension—see [BCHM06, Siu06] for the latest progress.

Proof of Proposition 4.1. Let $k \in \mathbb{N}^+$ be the index of $K_{Y_\lambda} + D_\lambda$, i.e., the smallest number such that $\mathcal{O}_{Y_\lambda}(k \cdot (K_{Y_\lambda} + D_\lambda)) \simeq \mathcal{O}_{Y_\lambda}$ and let $\gamma : \tilde{Y}_\lambda \rightarrow Y_\lambda$ be the associated index-one cover. We obtain that $K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda$ is a Cartier divisor for the trivial bundle, as claimed in (4.1.4).

The assertion that \tilde{Y}_λ is dlt follows from the definition and from the fact that discrepancies increase under finite morphisms, [KM98, 5.20]. If $y \in \tilde{Y}_\lambda$ is any point where $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is not log-smooth, then by the definition of dlt, the discrepancy of any divisor E that lies over y is $a(E, \tilde{Y}_\lambda, \tilde{D}_\lambda) > -1$. But since $K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda$ is Cartier, this number must be an integer, so $a(E, \tilde{Y}_\lambda, \tilde{D}_\lambda) \geq 0$. It follows that the pair $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is canonical at y , hence (4.1.5) is shown.

It remains to prove that $\kappa(\tilde{Y}, \tilde{D}) = 0$, as claimed in (4.1.6). Since $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is canonical wherever it is not log-smooth, the definition of canonical, [KM98, 2.26, 2.34], implies that $K_{\tilde{Y}} + \tilde{D}$ is represented by an effective, $\tilde{\lambda}$ -exceptional divisor, hence (4.1.6) follows. \square

Corollary 4.4. *Under the conditions of Proposition 4.1 further assume that $\dim Y = 2$. Then $(\tilde{Y}_\lambda, \tilde{D}_\lambda)$ is log-smooth along \tilde{D}_λ and \tilde{Y}_λ is \mathbb{Q} -factorial.*

Proof. The \mathbb{Q} -factoriality follows from (4.1.5) and [KM98, 4.11]. Log-smoothness follows from the classification of canonical surface singularities, [KM98, 4.5]. \square

4.B. The index-one cover in the presence of a Viehweg-Zuo sheaf. We will later consider the index-one cover in the presence of a Viehweg-Zuo sheaf \mathcal{A} . If $\kappa(\mathcal{A}) > 0$, we will show that \tilde{Y} is uniruled, and that the boundary cannot be empty. A similar line of argumentation was used in [KK05, KK07].

Proposition 4.5. *Under the conditions of Proposition 4.1 further assume that there exists a Viehweg-Zuo sheaf $\mathcal{A} \subset \mathrm{Sym}^n \Omega_Y^1(\log D)$ of positive Kodaira-Iitaka dimension, $\kappa(\mathcal{A}) > 0$. Then Y and \tilde{Y} are uniruled.*

The following—rather elementary—statements are used in the proof of Proposition 4.5. We formulate two separate lemmas for later reference.

Lemma 4.6. *Let (Y, D) be a log-smooth pair and assume that there exists a Viehweg-Zuo sheaf $\mathcal{A} \subset \mathrm{Sym}^n \Omega_Y^1(\log D)$. If $\lambda : Y \dashrightarrow Y_\lambda$ is a birational map whose inverse does not contract any divisor, Y_λ is normal and D_λ is the cycle-theoretic image of D , then there exists a Viehweg-Zuo sheaf $\mathcal{A}_\lambda \subset \mathrm{Sym}^{[n]} \Omega_{Y_\lambda}^1(\log D_\lambda)$ of Kodaira-Iitaka dimension $\kappa(\mathcal{A}_\lambda) \geq \kappa(\mathcal{A})$.*

Proof. The assumption that λ^{-1} does not contract any divisors and the normality of Y_λ guarantee that $\lambda^{-1} : Y_\lambda \dashrightarrow Y$ is well-defined and injective over an open subset $Y_\lambda^\circ \subset Y_\lambda$ whose complement has codimension $\mathrm{codim}_{Y_\lambda}(Y_\lambda \setminus Y_\lambda^\circ) \geq 2$. In particular, $D_\lambda|_{Y_\lambda^\circ} = (\lambda^{-1}|_{Y_\lambda^\circ})^{-1}D$. Let $\iota : Y_\lambda^\circ \hookrightarrow Y_\lambda$ denote the embedding and set $\mathcal{A}_\lambda := \iota_*((\lambda^{-1}|_{Y_\lambda^\circ})^{[*]}\mathcal{A})$. Fact 2.3 gives an inclusion $\mathcal{A}_\lambda \subset \mathrm{Sym}^{[n]} \Omega_{Y_\lambda}^1(\log D_\lambda)$. By construction $h^0(Y_\lambda, \mathcal{A}_\lambda^{[m]}) \geq h^0(Y, \mathcal{A}^{\otimes m})$ for all $m > 0$, hence $\kappa(\mathcal{A}_\lambda) \geq \kappa(\mathcal{A})$. \square

Lemma 4.7. *Under the conditions of Proposition 4.1 further assume that there exists a Viehweg-Zuo sheaf $\mathcal{A} \subset \mathrm{Sym}^n \Omega_Y^1(\log D)$. Then there exists a Viehweg-Zuo sheaf $\tilde{\mathcal{A}}_\lambda \subset \mathrm{Sym}^{[n]} \Omega_{\tilde{Y}_\lambda}^1(\log \tilde{D}_\lambda)$ of Kodaira-Iitaka dimension $\kappa(\tilde{\mathcal{A}}_\lambda) \geq \kappa(\mathcal{A})$.*

Proof. Let \mathcal{A}_λ be defined as in Lemma 4.6, and set $\tilde{\mathcal{A}}_\lambda := \gamma^{[*]}\mathcal{A}_\lambda$. The facts that $\tilde{\mathcal{A}}_\lambda$ is reflexive and that γ is étale imply that there exists an embedding $\tilde{\mathcal{A}}_\lambda \rightarrow \mathrm{Sym}^{[n]} \Omega_{\tilde{Y}_\lambda}^1(\log \tilde{D}_\lambda)$, as claimed. \square

Proof of Proposition 4.5. Since uniruledness is a birational property, and since images of uniruled varieties are again uniruled, it suffices to show the claim for \tilde{Y}_λ . We argue by contradiction and assume that \tilde{Y}_λ (and then also \tilde{Y}) is *not* uniruled—by [BDPP04, Cor. 0.3] this is equivalent to assuming that $K_{\tilde{Y}}$ is pseudo-effective. Again by [BDPP04, Thm. 0.2], this is in turn equivalent to the assumption that $K_{\tilde{Y}} \cdot C \geq 0$ for all moving curves $C \subset \tilde{Y}$.

As a first step, we will show that the assumption implies that the (Weil) divisor \tilde{D}_λ is zero. To this end, choose a polarization of \tilde{Y}_λ and consider a general complete intersection curve $\tilde{C}_\lambda \subset \tilde{Y}_\lambda$. Because \tilde{C}_λ is a complete intersection curve, it intersects the support of the effective divisor \tilde{D}_λ if the support is not empty. By general choice, the curve \tilde{C}_λ is contained in the smooth locus of \tilde{Y}_λ and avoids the indeterminacy locus of $\tilde{\lambda}^{-1}$. Its preimage $\tilde{C} := \tilde{\lambda}^{-1}(\tilde{C}_\lambda)$ is then a moving curve in \tilde{Y} which intersects \tilde{D} positively if and only if the Weil divisor \tilde{D}_λ is not zero. But

$$0 = (K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda) \cdot \tilde{C}_\lambda = (K_{\tilde{Y}} + \tilde{D}) \cdot \tilde{C} = \underbrace{K_{\tilde{Y}} \cdot \tilde{C}}_{\geq 0, \text{ as } \tilde{C} \text{ is moving}} + \underbrace{\tilde{D} \cdot \tilde{C}}_{\geq 0, \text{ as } \tilde{C} \text{ not in } \tilde{D}},$$

so $\tilde{D} \cdot \tilde{C} = 0$. In particular, \tilde{D}_λ is the zero divisor. This, combined with the fact that $\mathcal{O}_{\tilde{Y}_\lambda}(K_{\tilde{Y}_\lambda} + \tilde{D}_\lambda) \simeq \mathcal{O}_{\tilde{Y}_\lambda}$ implies that the canonical divisor $K_{\tilde{Y}_\lambda}$ is Cartier and its associated sheaf is trivial. In particular, the restrictions $\Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda}$ and $T_{\tilde{Y}_\lambda}|_{\tilde{C}_\lambda}$ are vector bundles of degree zero and so is the symmetric product $\text{Sym}^n \Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda}$.

Recall from Lemma 4.7 that there exists a Viehweg-Zuo sheaf of positive Kodaira-Iitaka dimension, say $\tilde{\mathcal{A}}_\lambda \subset \text{Sym}^{[n]} \Omega_{\tilde{Y}_\lambda}^1(\log \tilde{D}_\lambda)$. As \tilde{C}_λ is a general curve, the restriction $\tilde{\mathcal{A}}_\lambda|_{\tilde{C}_\lambda} \subset \text{Sym}^n \Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda}$ has positive degree. In particular, $\text{Sym}^n \Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda}$ is not semi-stable. Since symmetric products of semi-stable vector bundles are again semi-stable [HL97, Cor. 3.2.10], this implies that $\Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda}$ is likewise not semi-stable. Since $\deg \Omega_{\tilde{Y}_\lambda}^1|_{\tilde{C}_\lambda} = \deg T_{\tilde{Y}_\lambda}|_{\tilde{C}_\lambda} = 0$, this also implies that $T_{\tilde{Y}_\lambda}|_{\tilde{C}_\lambda}$ is not semi-stable.

In particular, the maximal destabilizing subsheaf of $T_{\tilde{Y}_\lambda}|_{\tilde{C}_\lambda}$ is of positive degree, hence ample. In this setup, a variant [KST07, Cor. 5] of Miyaoka's uniruledness criterion [Miy87, Cor. 8.6] applies to give the uniruledness of \tilde{Y}_λ . For more details on this criterion see the survey [KS06]. This ends the proof of Proposition 4.5. \square

Corollary 4.8. *Under the conditions of Proposition 4.5 the boundary divisor D_λ is not empty. In particular, D , \tilde{D}_λ and \tilde{D} are not empty.*

Proof. Again, we assume to the contrary that D_λ is empty. Proposition 4.1 then implies that $\kappa(\tilde{Y}) = 0$, while Proposition 4.5 asserts that \tilde{Y} is uniruled, a contradiction. \square

5. UNWINDING FAMILIES

We will consider projective families $g : Y \rightarrow T$ where the base T itself admits a fibration $\rho : T \rightarrow B$ such that g is isotrivial on all ρ -fibers. It is of course generally false that g would be the pull-back of a family defined over B . We will, however, show in this section that in some situations the family g does become a pull-back after a suitable base change.

We use the following notation for fibered products that appear in our setup.

Notation 5.1. Let T be a scheme, Y and Z schemes over T and $h : Y \rightarrow Z$ a T -morphism. If $t \in T$ is any point, let Y_t and Z_t denote the fibers of Y and Z over t . Furthermore, let h_t denote the restriction of h to Y_t . More generally, for any T -scheme \tilde{T} , let

$$h_{\tilde{T}} : \underbrace{Y \times_T \tilde{T}}_{=: Y_{\tilde{T}}} \rightarrow \underbrace{Z \times_T \tilde{T}}_{=: Z_{\tilde{T}}}$$

denote the pull-back of h to \tilde{T} . The situation is summarized in the following commutative diagram.

$$\begin{array}{ccccc}
 Y_{\tilde{T}} & \xrightarrow{h_{\tilde{T}}} & Z_{\tilde{T}} & & Y & \xrightarrow{h} & Z \\
 & \searrow & \swarrow & & \searrow & & \swarrow \\
 & & \tilde{T} & \xrightarrow{\quad} & T & &
 \end{array}$$

The setup of the current section is then formulated as follows.

Assumption 5.2. *Throughout the present section, consider a sequence of morphisms between algebraic varieties,*

$$Y \xrightarrow[\text{smooth, projective}]{g} T \xrightarrow[\text{smooth, rel. dim.}=1]{\rho} B,$$

where g is a smooth projective family and ρ is smooth of relative dimension 1, but not necessarily projective. Assume further that for all $b \in B$, there exists a smooth variety F_b such that for all $t \in T_b$, there exists an isomorphism $Y_t \simeq F_b$.

5.A. Relative isomorphisms of families over the same base. To start, recall the well-known fact that an isotrivial family of varieties of general type over a curve becomes trivial after passing to an étale cover of the base. As we are not aware of an adequate reference, we include a proof here.

Lemma 5.3. *Let $b \in B$ and assume that $\text{Aut}(F_b)$ is finite. Then the natural morphism $\iota : I = \text{Isom}_{T_b}(Y_b, T_b \times F_b) \rightarrow T_b$ is finite and étale. Furthermore, pull-back to I yields an isomorphism of I -schemes $Y_I \simeq I \times F_b$.*

Proof. Consider the T_b -scheme

$$H := \text{Hilb}_{T_b}(Y_b \times_{T_b} (T_b \times F_b)) \simeq \text{Hilb}_{T_b}(Y_b \times F_b).$$

By Assumption 5.2, $H_t \simeq \text{Hilb}(F_b \times F_b)$ for all $t \in T_b$. Similarly, $I_t \simeq \text{Aut}(F_b)$, hence I is one-dimensional and $\text{length}(I_t)$ is constant on T_b . Since I is open in H , the union of components of H that contain I , denoted by H^I , is also one-dimensional.

Recall that $H \rightarrow T_b$ is projective, so $H^I \rightarrow T_b$ is also projective, hence finite. Since $H \rightarrow T_b$ is flat, $\text{length}(H_t)$ is constant. Furthermore, $I \subseteq H^I$ is open, so $H_t^I = I_t$ and hence $\text{length}(H_t) = \text{length}(I_t)$ for a general $t \in T_b$. However, we observed above that $\text{length}(I_t)$ is also constant, so we must have that $\text{length}(H_t) = \text{length}(I_t)$ for all $t \in T_b$, and since $I \subseteq H^I$, this means that $I = H^I$ and $\iota : I \rightarrow T_b$ is finite and unramified, hence étale.

In order to prove the global triviality of Y_I , consider $\text{Isom}_I(Y_I, I \times F_b)$. Recall that taking Hilb and Isom commutes with base change, and so we obtain an isomorphism

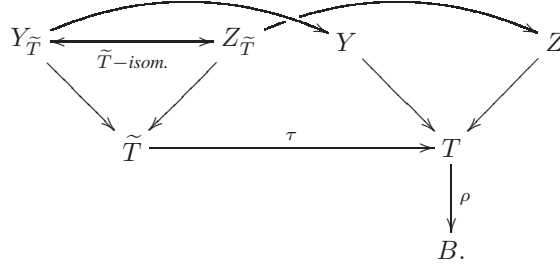
$$\text{Isom}_I(Y_I, I \times F_b) \simeq I \times_{T_b} \text{Isom}_{T_b}(Y_b, T_b \times F_b) \simeq I \times_{T_b} I.$$

This scheme admits a natural section over T_b , namely its diagonal, which induces an I -isomorphism between Y_I and $I \times F_b$. \square

The preceding Lemma 5.3 can be used to compare two families whose associated moduli maps agree. We show that in our setup any two such families become globally isomorphic after changing base.

Lemma 5.4. *In addition to Assumption 5.2, assume that there exists another projective morphism, $Z \rightarrow T$, with the following property: for any $b \in B$ and any $t \in T_b$, we have $Y_t \simeq Z_t \simeq F_b$. Then*

(5.4.1) *there exists a surjective morphism $\tau : \tilde{T} \rightarrow T$ such that the pull-back families of Y and Z to \tilde{T} are isomorphic as \tilde{T} -schemes, i.e., we have a commutative diagram as follows:*



Furthermore, if for all $b \in B$, the group $\text{Aut}(F_b)$ is finite, then \tilde{T} can be chosen such that the following holds. Let $\tilde{T}' \subset \tilde{T}$ be any irreducible component. Then

- (5.4.2) τ is quasi-finite,
- (5.4.3) the image set $\tau(\tilde{T}')$ is a union of ρ -fibers, and
- (5.4.4) if \tilde{T}' dominates B , then there exists an open subset $B^\circ \subset (\rho \circ \tau)(\tilde{T}')$ such that $\tau|_{\tilde{T}'}$ is finite and étale over B° . More precisely, if we set $T^\circ := \rho^{-1}(B^\circ)$ and $\tilde{T}^\circ := \tau^{-1}(T^\circ) \cap \tilde{T}'$, then the restriction $\tau|_{\tilde{T}^\circ} : \tilde{T}^\circ \rightarrow T^\circ$ is finite and étale.

Remark 5.4.5. In Lemma 5.4 we do not claim that \tilde{T} is irreducible or connected.

Proof of Lemma 5.4. Set $\tilde{T} := \text{Isom}_T(Y, Z)$ and let $\tau : \tilde{T} \rightarrow T$ be the natural morphism. Again, taking Isom commutes with base change, and we have an isomorphism $\tilde{T} \times_T \tilde{T} \simeq \text{Isom}_{\tilde{T}}(Y_{\tilde{T}}, Z_{\tilde{T}})$. Similarly, for all $b \in B$, and for all $t \in T_b$, there is a natural one-to-one correspondence between \tilde{T}_t and $\text{Aut}(F_b)$. In particular, we obtain that τ is surjective. As before, observe that $\tilde{T} \times_T \tilde{T}$ admits a natural section, the diagonal. This shows (5.4.1).

If for all $b \in B$, $\text{Aut}(F_b)$ is finite, then the restriction of τ to any ρ -fiber, $\tau_b : \tilde{T}_b \rightarrow T_b$ is finite étale by Lemma 5.3. This shows (5.4.2) and (5.4.3). Furthermore, it implies that if $\tilde{T}' \subset \tilde{T}$ is a component that dominates B , neither the ramification locus of $\tau|_{\tilde{T}'}$, nor the locus where $\tau|_{\tilde{T}'}$ is not finite dominates B . In fact, if we let B° denote the open set of B where $\#\text{Aut}(F_b)$ is constant, then (5.4.4) holds for B° . \square

5.B. Families where ρ has a section. Now consider Assumption 5.2 in case the morphism ρ admits a section $\sigma : B \rightarrow T$ such that $Z = Y_B \times_B T$. As a corollary to Lemma 5.4, we will show that in this situation \tilde{T} always contains a component \tilde{T}' such that the pull-back family $Y_{\tilde{T}'}$ comes from B . Better still, the restriction $\tau|_{\tilde{T}'}$ is “relatively étale” in the sense that $\tau|_{\tilde{T}'}$ is étale and that $\rho \circ \tau|_{\tilde{T}'}$ has connected fibers.

Corollary 5.5. *Under the conditions of Lemma 5.4 assume that ρ admits a section $\sigma : B \rightarrow T$, and that $Z = Y_B \times_B T$. Then there exists an irreducible component $\tilde{T}' \subset \tilde{T}$ such that*

- (5.5.1) \tilde{T}' surjects onto B , and
- (5.5.2) the restricted morphism $\rho \circ \tau|_{\tilde{T}'} : \tilde{T}' \rightarrow B$ has connected fibers.

Proof. It is clear from the construction that $Y_B \simeq Z_B$. This isomorphism corresponds to a morphism $\tilde{\sigma} : B \rightarrow \text{Isom}_T(Y, Z) = \tilde{T}$. Let $\tilde{T}' \subset \tilde{T}$ be an irreducible component that contains the image of $\tilde{\sigma}$. The existence of a section guarantees that $\rho \circ \tau|_{\tilde{T}'} : \tilde{T}' \rightarrow B$ is surjective and its fibers are connected. \square

One particular setup where a section is known to exist is when T is a birationally ruled surface over B . The following will become important later.

Corollary 5.6. *In addition to Assumption 5.2, suppose that B is a smooth curve and that the general ρ -fiber is isomorphic to \mathbb{P}^1 , \mathbb{A}^1 or $(\mathbb{A}^1)^* = \mathbb{A}^1 \setminus \{0\}$. Then there exist non-empty Zariski open sets $B^\circ \subset B$, $T^\circ := \rho^{-1}(B^\circ)$ and a commutative diagram*

$$\begin{array}{ccc} \tilde{T}^\circ & \xrightarrow[\text{\textit{étale}}]{\tau} & T^\circ \\ & \searrow \text{\textit{conn. fibers}} & \downarrow \rho \\ & & B^\circ \end{array}$$

such that

(5.6.1) *the fibers of $\rho \circ \tau$ are again isomorphic to \mathbb{P}^1 , \mathbb{A}^1 or $(\mathbb{A}^1)^*$, respectively, and*

(5.6.2) *the pull-back family $Y_{\tilde{T}^\circ}$ comes from B° , i.e., there exists a projective family $Z \rightarrow B^\circ$ and a \tilde{T}° -isomorphism*

$$Y_{\tilde{T}^\circ} \simeq Z_{\tilde{T}^\circ}.$$

Remark 5.6.3. If the general ρ -fiber is isomorphic to \mathbb{P}^1 or \mathbb{A}^1 , the morphism τ is necessarily an isomorphism. Shrinking B° further, if necessary, $\rho : T^\circ \rightarrow B^\circ$ will then even be a trivial \mathbb{P}^1 - or \mathbb{A}^1 -bundle, respectively.

Proof. Shrinking B , if necessary, we may assume that all ρ -fibers are isomorphic to \mathbb{P}^1 , \mathbb{A}^1 or $(\mathbb{A}^1)^*$, and hence that T is smooth. Then it is always possible to find a relative smooth compactification of T , i.e. a smooth B -variety $\overline{T} \rightarrow B$ and a smooth divisor $D \subset T$ such that $\overline{T} \setminus D$ and T are isomorphic B -schemes.

By Tsen's theorem, [Sha94, p. 73], there exists a section $\sigma : B \rightarrow \overline{T}$. In fact, there exists a positive dimensional family of sections, so that we may assume without loss of generality that $\sigma(B)$ is not contained in D .

Let $B^\circ \subset B$ be the open subset such that for all $b \in B^\circ$, $\overline{T}_b \simeq \mathbb{P}^1$, T_b is isomorphic to \mathbb{P}^1 , \mathbb{A}^1 or $(\mathbb{A}^1)^*$, respectively, and $\sigma(b) \notin D$. Using that any connected finite étale cover of T_b is again isomorphic to T_b , and shrinking B° further, Corollary 5.5 yields the claim. \square

Remark 5.7. Throughout the article we work over the field of complex numbers \mathbb{C} , thus we kept that assumption here as well. However, we would like to note that the results of this section work over an arbitrary algebraically closed base field k .

PART II. PROOF OF THEOREM 1.1

6. SETUP AND NOTATION

The cases $\kappa(S^\circ) = -\infty, 0$ and 1 are considered separately in Sections 7–9 below.

The following setup and notation will be used throughout the rest of the article: As in Theorem 1.1, we fix a smooth compactification $S^\circ \subset S$ such that $D := S \setminus S^\circ$ is a divisor with simple normal crossings. The log minimal model program then yields a birational morphism $\lambda : S \rightarrow S_\lambda$, with the following properties.

(6.0.1) The surface S_λ is normal and \mathbb{Q} -factorial.

(6.0.2) If D_λ is the cycle-theoretic image of D , then (S_λ, D_λ) is a reduced dlt pair.

(6.0.3) By Lemma 2.9 and Theorem 2.10, the extension theorem holds for (S_λ, D_λ) .

Again, recall from [VZ02, Thm. 1.4] that there exists a Viehweg-Zuo sheaf $\mathcal{A} \subset \text{Sym}^n \Omega_S^1(\log D)$ of positive Kodaira-Iitaka dimension $\kappa(\mathcal{A}) \geq \text{Var}(f^\circ) > 0$. By Lemma 4.6, there exists a Viehweg-Zuo sheaf $\mathcal{A}_\lambda \subset \text{Sym}^{[n]} \Omega_{S_\lambda}^1(\log D_\lambda)$ of Kodaira-Iitaka dimension $\kappa(\mathcal{A}_\lambda) \geq \kappa(\mathcal{A})$.

7. PROOF IN CASE $\kappa(S^\circ) = -\infty$

In this case, the log canonical bundle $K_S + D$ has negative Kodaira-Iitaka dimension, and (S_λ, D_λ) is a pair that either has the structure of a Mori-Fano fiber space or is a log-Fano pair with Picard number $\rho(S_\lambda) = 1$. However, since the extension theorem holds, Theorem 3.1 rules out the case that $\rho(S_\lambda) = 1$. The pair (S_λ, D_λ) thus always admits a fibration, independently of the choices made in its construction. In particular, there exists a smooth curve C and a fibration $\pi_\lambda : S_\lambda \rightarrow C$ with connected fibers, such that $-(K_{S_\lambda} + D_\lambda)$ intersects the general fiber positively.

Setting $\pi := \pi_\lambda \circ \lambda$, the general fiber F of π is then a rational curve that intersects the boundary in one point, if at all. In particular, the restriction of the family f° to $F \cap S^\circ$ is necessarily isotrivial by [Kov00]. The detailed description of the moduli map in case $\kappa(S^\circ) = -\infty$ then follows from Corollary 5.6 and Remark 5.6.3. \square

8. PROOF THAT $\kappa(S^\circ) \neq 0$

8.A. Setup. To prove Theorem 1.1 in this case, we argue by contradiction and assume that $\kappa(S^\circ) = 0$. Let $(\tilde{S}_\lambda, \tilde{D}_\lambda)$ be the index-one cover of a log-minimal model, as in Proposition 4.1. The main properties of $(\tilde{S}_\lambda, \tilde{D}_\lambda)$ are summarized as follows.

- (8.1.1) The pair $(\tilde{S}_\lambda, \tilde{D}_\lambda)$ is \mathbb{Q} -factorial and dlt (Proposition 4.1 and Corollary 4.4).
- (8.1.2) There exists a Viehweg-Zuo sheaf $\tilde{\mathcal{A}}_\lambda \subset \text{Sym}^{[n]} \Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)$ of positive Kodaira-Iitaka dimension (Lemma 4.7).
- (8.1.3) If $\tilde{S}_{\lambda, \text{reg}} \subset \tilde{S}_\lambda$ is the maximal smooth open subset, then the restriction $\tilde{\mathcal{A}}_\lambda|_{\tilde{S}_{\lambda, \text{reg}}}$ is invertible ([OSS80, 1.1.15 on p. 154]).
- (8.1.4) \tilde{S}_λ is uniruled, and the boundary \tilde{D}_λ is not empty (Proposition 4.5 and Corollary 4.8).
- (8.1.5) The divisor $K_{\tilde{S}_\lambda} + \tilde{D}_\lambda$ is Cartier and trivial (Proposition 4.1).
- (8.1.6) If $p \in \tilde{D}_\lambda$ is any point, then $(\tilde{S}_\lambda, \tilde{D}_\lambda)$ is log-smooth at p . In particular, $\Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)$ is locally free along \tilde{D}_λ (Corollary 4.4).

8.B. Outline of the proof. As a first step in the proof of Theorem 1.1, we aim to apply Theorem 3.1, in order to show that \tilde{S}_λ is fibered over a curve, with rational fibers that intersect the boundary twice. Since Theorem 3.1 works best in the case $\kappa = -\infty$ we need to decrease the boundary coefficients slightly and perform extra contractions before Theorem 3.1 can be applied to prove the existence of a fibration.

The fiber space structure of \tilde{S}_λ is then used to analyze the restriction of the Viehweg-Zuo sheaf $\tilde{\mathcal{A}}_\lambda$ to a suitable boundary component $\tilde{D}'_\lambda \subset \tilde{D}_\lambda$. Even though there is no smooth family over \tilde{D}'_λ , it will turn out that the restriction $\tilde{\mathcal{A}}_\lambda|_{\tilde{D}'_\lambda}$ can be interpreted as a Viehweg-Zuo sheaf on \tilde{D}'_λ , which again has positive Kodaira-Iitaka dimension. This leads to contradiction and thus finishes the proof.

8.C. Minimal models of $(\tilde{S}_\lambda, \tilde{D}_\lambda)$. Since \tilde{D}_λ is not empty and $K_{\tilde{S}_\lambda} \equiv -\tilde{D}_\lambda$, it follows that for any rational number $0 < \varepsilon < 1$,

$$\kappa(\tilde{S}_\lambda, (1 - \varepsilon)\tilde{D}_\lambda) = \kappa(K_{\tilde{S}_\lambda} + (1 - \varepsilon)\tilde{D}_\lambda) = \kappa(\varepsilon \cdot K_{\tilde{S}_\lambda}) = \kappa(\tilde{S}_\lambda) = -\infty.$$

Choose a rational number $0 < \varepsilon < 1$ and perform a minimal model program for the pair $(\tilde{S}_\lambda, (1 - \varepsilon)\tilde{D}_\lambda)$. This will produce a birational morphism $\mu : \tilde{S}_\lambda \rightarrow S_\mu$. Let D_μ be the cycle-theoretic image of \tilde{D}_λ . Since $(\tilde{S}_\lambda, (1 - \varepsilon)\tilde{D}_\lambda)$ has dlt singularities, the pair $(S_\mu, (1 - \varepsilon)D_\mu)$ will also be dlt, in fact, it will be klt.

Remark 8.2. Since $\kappa(\tilde{S}_\lambda, (1 - \varepsilon)\tilde{D}_\lambda) = -\infty$, either $\rho(S_\mu) > 1$ and the pair $(S_\mu, (1 - \varepsilon)D_\mu)$ is a Mori-Fano fiber space, or $\rho(S_\mu) = 1$.

Remark 8.3. It follows from the equation $K_{\tilde{S}_\lambda} \equiv -\tilde{D}_\lambda$ that for any $0 < \varepsilon', \varepsilon'' < 1$, the divisors $K_{\tilde{S}_\lambda} + (1 - \varepsilon')\tilde{D}_\lambda$ and $K_{\tilde{S}_\lambda} + (1 - \varepsilon'')\tilde{D}_\lambda$ are multiples of one another. In particular, the birational morphism μ is a minimal model program for the pair $(\tilde{S}_\lambda, (1 - \varepsilon)\tilde{D}_\lambda)$, independently of the chosen $0 < \varepsilon < 1$. It follows that $(S_\mu, (1 - \varepsilon)D_\mu)$ has dlt singularities for all ε . In particular, it follows directly from the definition of discrepancy [KM98, 2.26] that the reduced pair (S_μ, D_μ) is dlc in the sense of Definition 2.7.

8.D. The fiber space structure of \tilde{S}_λ . We apply Theorem 3.1 in order to show that S_μ is a fiber space.

Proposition 8.4. *One has that $\rho(S_\mu) > 1$. In particular, S_μ has the structure of a non-trivial Mori-Fano fiber space.*

Proof. If $-(K_{S_\mu} + D_\mu)$ is not ample, then $\rho(S_\mu) > 1$ and the statement follows from Remark 8.2. If $-(K_{S_\mu} + D_\mu)$ is ample, then Remark 8.3 and Example 2.9 imply that (S_μ, D_μ) is finitely dominated by smooth analytic pairs. Then by Theorem 2.10, the extension theorem holds for (S_μ, D_μ) . According to Lemma 4.6 there exists a Viehweg-Zuo subsheaf $\mathcal{A}_\mu \subset \text{Sym}^{[n]} \Omega_{S_\mu}^1(\log D_\mu)$ of positive Kodaira-Iitaka dimension and then Theorem 3.1 and Remark 8.2 imply the desired statement. \square

Corollary 8.5. *There exists a morphism $\pi : \tilde{S}_\lambda \rightarrow C$ to a smooth curve, and an open set $C^\circ \subset C$ such that for any $c \in C^\circ$, the associated fiber $F_c := \pi^{-1}(c)$ is a smooth rational curve which is entirely contained in the log-smooth locus $(\tilde{S}_\lambda, \tilde{D}_\lambda)_{\text{reg}}$ and which intersects the boundary \tilde{D}_λ transversally in exactly two points. In particular, the sheaf $\Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)|_{F_c}$ is trivial.*

Proof. The existence of π and the rationality of the general fiber follows from Proposition 8.4. The number of intersection points follows from $K_{\tilde{S}_\lambda} + \tilde{D}_\lambda \equiv 0$. The triviality of $\Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)|_{F_c}$ follows from standard sequences, see [KK05, 2.14] and (8.9.1) below. \square

Corollary 8.6. *If $c \in C^\circ$ is a general point, then the restriction $\tilde{\mathcal{A}}_\lambda|_{F_c}$ is trivial.*

Proof. Since F_c is a general fiber, $\tilde{\mathcal{A}}_\lambda^{[r]}|_{F_c}$ is an invertible sheaf for any $r \in \mathbb{Z}$ by (8.1.3). In particular, $\tilde{\mathcal{A}}_\lambda^{[r]}|_{F_c} \simeq (\tilde{\mathcal{A}}_\lambda|_{F_c})^{\otimes r}$. Fix an $r \in \mathbb{N}$ such that $h^0(\tilde{S}_\lambda, \tilde{\mathcal{A}}_\lambda^{[r]}) > 0$. Then there exists a non-trivial and hence injective morphism

$$\tilde{\mathcal{A}}_\lambda^{[r]}|_{F_c} \rightarrow (\text{Sym}^{[r \cdot n]} \Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda))|_{F_c} \simeq \text{Sym}^{r \cdot n}(\Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)|_{F_c}).$$

The triviality of the sheaf $\Omega_{\tilde{S}_\lambda}^1(\log \tilde{D}_\lambda)|_{F_c}$ implies that $\deg(\tilde{\mathcal{A}}_\lambda^{[r]}|_{F_c}) \leq 0$. Since F_c passes through a general point of \tilde{D}_λ , a general section of the sheaf $\tilde{\mathcal{A}}_\lambda^{[r]}$ does not vanish along all of F_c . Therefore $\tilde{\mathcal{A}}_\lambda^{[r]}|_{F_c}$ is a line bundle of non-positive degree that has a global section. Consequently it is trivial. Since $F_c \simeq \mathbb{P}^1$, this implies the statement. \square

8.E. Non-triviality of $\tilde{\mathcal{A}}_\lambda|_D$. Now consider a section $\sigma \in H^0(\tilde{S}_\lambda, \tilde{\mathcal{A}}_\lambda^{[r]})$, let F_c be a general π -fiber and $y \in F_c$ a general point of F_c . The triviality of $\tilde{\mathcal{A}}_\lambda^{[r]}$ on F_c can now be used to compare the value of σ at a y with its value at a point where F_c hits the boundary \tilde{D}_λ . It will follow that σ is completely determined by the values it takes on the boundary.

Lemma 8.7. *There exists an irreducible component $\tilde{D}'_\lambda \subset \tilde{D}_\lambda$ such that for any $r \in \mathbb{N}$, the natural restriction morphism*

$$H^0(\tilde{S}_\lambda, \tilde{\mathcal{A}}_\lambda^{[r]}) \rightarrow H^0(\tilde{D}'_\lambda, \tilde{\mathcal{A}}_\lambda^{[r]}|_{\tilde{D}'_\lambda})$$

is injective. In particular, the restriction $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda}$ is a non-trivial invertible sheaf and its Kodaira-Iitaka dimension equals the Kodaira-Iitaka dimension of $\widetilde{\mathcal{A}}_\lambda$, i.e., $\kappa(\widetilde{\mathcal{A}}_\lambda) = \kappa(\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda})$.

Proof. Corollary 8.5 implies that there exists a component $\widetilde{D}'_\lambda \subset \widetilde{D}_\lambda$ that is hit by all the curves $(F_c)_{c \in C^\circ}$. Now let r be any given number. If $h^0(\widetilde{S}_\lambda, \widetilde{\mathcal{A}}_\lambda^{[r]}) = 0$, there is nothing to show. Otherwise, using the notation of Corollary 8.5, set

$$\widetilde{D}'_{\lambda,\circ} := \widetilde{D}'_\lambda \cap \pi^{-1}(C^\circ).$$

Since a section in the trivial bundle is determined by its value at any given point, a section $\sigma \in H^0(\widetilde{S}_\lambda, \widetilde{\mathcal{A}}_\lambda^{[r]})$ is uniquely determined by its restriction to $\widetilde{D}'_{\lambda,\circ} \subset \widetilde{D}'_\lambda$ by Corollary 8.6. Finally, note that $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda}$ is invertible by (8.1.3) and (8.1.6). Thus the claim is shown. \square

Remark 8.8. Let $\iota : \widetilde{\mathcal{A}}_\lambda \rightarrow \mathrm{Sym}^{[n]} \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)$ denote the injection of our Viehweg-Zuo sheaf into the sheaf of pluri-log differentials. Then its restriction $\iota|_{\widetilde{D}'_\lambda}$ is injective.

8.F. Existence of pluri-forms on \widetilde{D}'_λ . As a last ingredient in the proof, we show that sections in tensor powers of the invertible sheaf $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda}$ can again be interpreted as pluri-forms on the boundary.

Lemma 8.9. *Let $\widetilde{D}''_\lambda = (\widetilde{D}_\lambda - \widetilde{D}'_\lambda)|_{\widetilde{D}'_\lambda}$. Then there exists a number $m \leq n$ and an injective sheaf morphism $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda} \rightarrow \mathrm{Sym}^m \Omega_{\widetilde{D}'_\lambda}^1(\log \widetilde{D}''_\lambda)$.*

Proof. Since $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda}$ is invertible by (8.1.3) and (8.1.6), it is enough to show that there exists a non-zero morphism $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda} \rightarrow \mathrm{Sym}^m \Omega_{\widetilde{D}'_\lambda}^1(\log \widetilde{D}''_\lambda)$, for some $m \in \mathbb{N}$. We will use the following sequence that relates restrictions of log-forms with log-forms on the restriction—the sequence is discussed in [KK05, 2.13].

$$(8.9.1) \quad 0 \longrightarrow \Omega_{\widetilde{D}'_\lambda}^1(\log \widetilde{D}''_\lambda) \xrightarrow{\alpha} \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)|_{\widetilde{D}'_\lambda} \xrightarrow{\beta} \mathcal{O}_{\widetilde{D}'_\lambda} \longrightarrow 0.$$

Along with this sequence comes the standard filtration of the symmetric product,

$$\mathrm{Sym}^n \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)|_{\widetilde{D}'_\lambda} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \dots \supseteq \mathcal{F}^n \supseteq \mathcal{F}^{n+1} = 0,$$

with quotients

$$(8.9.2) \quad 0 \longrightarrow \mathcal{F}^{p+1} \xrightarrow{\alpha_p} \mathcal{F}^p \xrightarrow{\beta_p} \mathrm{Sym}^p \Omega_{\widetilde{D}'_\lambda}^1(\log \widetilde{D}''_\lambda) \longrightarrow 0.$$

See [Har77, ex. II.5.16] for details. As in Remark 8.8, let ι be the injection of the Viehweg-Zuo sheaf $\widetilde{\mathcal{A}}_\lambda$ into the sheaf of pluri-log differentials $\mathrm{Sym}^{[n]} \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)$. Recall from Lemma 8.7 that $\widetilde{\mathcal{A}}_\lambda|_{\widetilde{D}'_\lambda}$ has positive Kodaira-Iitaka dimension and from Remark 8.8 that it embeds into $\mathrm{Sym}^{[n]} \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)|_{\widetilde{D}'_\lambda} \simeq \mathrm{Sym}^n \Omega_{\widetilde{S}_\lambda}^1(\log \widetilde{D}_\lambda)|_{\widetilde{D}'_\lambda}$.

First consider the sequence in (8.9.2) for $p = 0$. Since $\mathrm{Sym}^0 \Omega_{\widetilde{D}'_\lambda}^1(\log \widetilde{D}''_\lambda) = \mathcal{O}_{\widetilde{D}'_\lambda}$, and since any morphism from an invertible sheaf of positive Kodaira-Iitaka dimension to the structure sheaf is necessarily zero, the composition $\beta_0 \circ \iota|_{\widetilde{D}'_\lambda}$ is zero, and the restriction

$\iota|_{\widetilde{D}'_\lambda}$ factors via an injection $\iota_1 : \widetilde{\mathcal{A}}|_{\widetilde{D}'_\lambda} \rightarrow \mathcal{F}^1$.

Next consider (8.9.2) for $p = 1$. If $\beta_1 \circ \iota_1$ is non-zero, the proof is finished. Otherwise, ι_1 factors via an injection $\iota_2 : \widetilde{\mathcal{A}}|_{\widetilde{D}'_\lambda} \rightarrow \mathcal{F}^2$, and we consider (8.9.2) for $p = 2$, etc. This process must stop after no more than n steps. Thus the claim is shown. \square

8.G. End of the proof. Using the notation introduced in Lemma 8.9, the adjunction formula shows that $K_{\tilde{D}'_\lambda} + \tilde{D}''_\lambda = (K_{\tilde{S}_\lambda} + \tilde{D}_\lambda)|_{\tilde{D}'_\lambda} \equiv 0$. In particular, $\deg \Omega^1_{\tilde{D}'_\lambda}(\log \tilde{D}''_\lambda) = 0$. On the other hand, Lemma 8.9 asserts the existence of an injective morphism of sheaves $\tilde{\mathcal{A}}_\lambda|_{\tilde{D}'_\lambda} \rightarrow \mathrm{Sym}^m \Omega^1_{\tilde{D}'_\lambda}(\log \tilde{D}''_\lambda)$. By Lemma 8.7, $\tilde{\mathcal{A}}_\lambda|_{\tilde{D}'_\lambda}$ has positive Kodaira-Iitaka dimension $\kappa(\tilde{\mathcal{A}}_\lambda|_{\tilde{D}'_\lambda}) = \kappa(\tilde{\mathcal{A}}_\lambda) > 0$. This is clearly absurd, and the proof of Theorem 1.1 is thus finished in the case $\kappa(S^\circ) = 0$. \square

9. PROOF IN CASE $\kappa(S^\circ) = 1$

In this case the statements of Theorem 1.1 follow from the results of Section 5 when one applies the logarithmic minimal model program. The following proposition summarizes the standard description of surfaces with logarithmic Kodaira dimension 1.

Proposition 9.1. *If $\kappa(S^\circ) = 1$, then there exists a smooth curve C and a fibration $\pi : S \rightarrow C$ with connected fibers, such that $K_S + D$ is trivial on the general fiber. In particular, one of the following holds:*

- (9.1.1) *The general fiber is an elliptic curve and no component of D dominates C , or*
- (9.1.2) *The general fiber is isomorphic to \mathbb{P}^1 and D intersects the general fiber in exactly two points.*

Proof. The logarithmic abundance theorem in dimension 2, see e.g. [KM98, 3.3], asserts that for $n \gg 0$ the linear system $|n(K_{S_\lambda} + D_\lambda)|$ yields a morphism to a curve $\pi_\lambda : S_\lambda \rightarrow C$, such that $K_{S_\lambda} + D_\lambda$ is trivial on the general fiber F_λ of π_λ . Likewise, if $\pi := \pi_\lambda \circ \phi$ and $F \subset S$ is a general fiber of π , then $K_S + D$ is trivial on F . Statements (9.1.1) and (9.1.2) describe the only two ways this can happen. \square

To finish the proof of Theorem 1.1, consider the morphism $\pi : S \rightarrow C$ provided by Proposition 9.1. Let $V \subseteq C$ be the locus over which π is smooth and either $D \cap \pi^{-1}(V) = \emptyset$ or $\pi|_D$ is étale. Consider the restriction of π to $U := \pi^{-1}(V) \cap S^\circ$. By Proposition 9.1, the general fiber of $\pi|_U$ is either an elliptic curve, or it is isomorphic to \mathbb{C}^* . In both cases, it follows from [Kov96] and [Kov00] that f is isotrivial on the fibers of $\pi : U \rightarrow V$. The factorization of the moduli map follows.

It remains to give the detailed description of the moduli map. If the general fibers of π are isomorphic to \mathbb{C}^* , Corollary 5.6 yields the claim. Otherwise, take an irreducible multisection $\hat{V} \subset S$, restrict V further if necessary so \hat{V} is étale over V and take a base change to \hat{V} . We end up with a section $\sigma : \hat{V} \rightarrow \hat{U} := U \times_V \hat{V}$. Finally, set $\hat{X} := X \times_U \hat{U}$, and $Z := \hat{V} \times_\sigma \hat{X}$. Shrinking V further, if necessary, an application of Lemma 5.4 completes the proof of Theorem 1.1. \square

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